Student's Name: ______________________

Student Number:

Teacher's Name: ______________________

ABBOTSLEIGH

2023

HIGHER SCHOOL CERTIFICATE

Assessment 4

Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen.
- **NESA approved** calculators may be used.
- **NESA approved** reference sheet is provided.
- All necessary working should be shown in every question.
- Write your NESA number on each page.
- Start a new page for each of Questions 11 - 14.
- Answer the Multiple Choice questions on the answer sheet provided.
- If you do not attempt a whole question, you must write the question number at the top of a blank page with your NESA number and the words **'NOT ATTEMPTED'** written clearly on it.

Total marks - 70

- Attempt Sections I and II.

Section I   Pages 3 - 7

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II   Pages 8- 16

60 marks

- Attempt Questions 11 – 14.
- Allow about 1 hour and 45 minutes for this section.
- All questions are of equal value.

Outcomes to be assessed:

Year 11 Mathematics Extension 1 outcomes

A student:

ME11-1

uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses

ME11-2

manipulates algebraic expressions and graphical functions to solve problems

ME11-3

applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems

ME11-4

applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change

ME11-5

uses concepts of permutations and combinations to solve problems involving counting or ordering

ME11-7

communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Year 12 Mathematics Extension 1 outcomes

A student:

ME12-1

applies techniques involving proof or calculus to model and solve problems

ME12-2

applies concepts and techniques involving vectors and projectiles to solve problems

ME12-3

applies advanced concepts and techniques in simplifying expressions involving compound angles and solving trigonometric equations

ME12-4

uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

ME12-7

evaluates and justifies conclusions, communicating a position clearly in appropriate mathematical forms

SECTION I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 - 10

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1. Given $f(x)=\sqrt{x}-3$, what are the domain and range of $f^{-1}(x)$?

   (A) $x \geq -3,\ \ y \geq 0$

   (B) $x \geq -3,\ \ y \geq -3$

   (C) $x \geq 0,\ \ y \geq 0$

   (D) $x \geq 0,\ \ y \geq -3$

2. Given $\tan\theta = \dfrac{1}{3}$, what is the exact value of $\tan\left(\theta+\dfrac{\pi}{3}\right)$?

   (A) $\dfrac{\sqrt{3}+3}{3\sqrt{3}-1}$

   (B) $\dfrac{\sqrt{3}-3}{3\sqrt{3}+1}$

   (C) $\dfrac{1+3\sqrt{3}}{3-\sqrt{3}}$

   (D) $\dfrac{1-3\sqrt{3}}{3+\sqrt{3}}$

Section I continues on next page

3. What is the domain and range of the function $y = 6\sin^{-1}(3x)$?

(A) Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[-6\pi, 6\pi]$.

(B) Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[-3\pi, 3\pi]$.

(C) Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[0, 6\pi]$.

(D) Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[0, 3\pi]$.

4. Layla projects an arrow at an angle of $45°$ to the horizontal with an initial velocity of 50 ms^{-1}. What is the horizontal speed of the arrow?

(A) $\sqrt{2}$ ms^{-1}

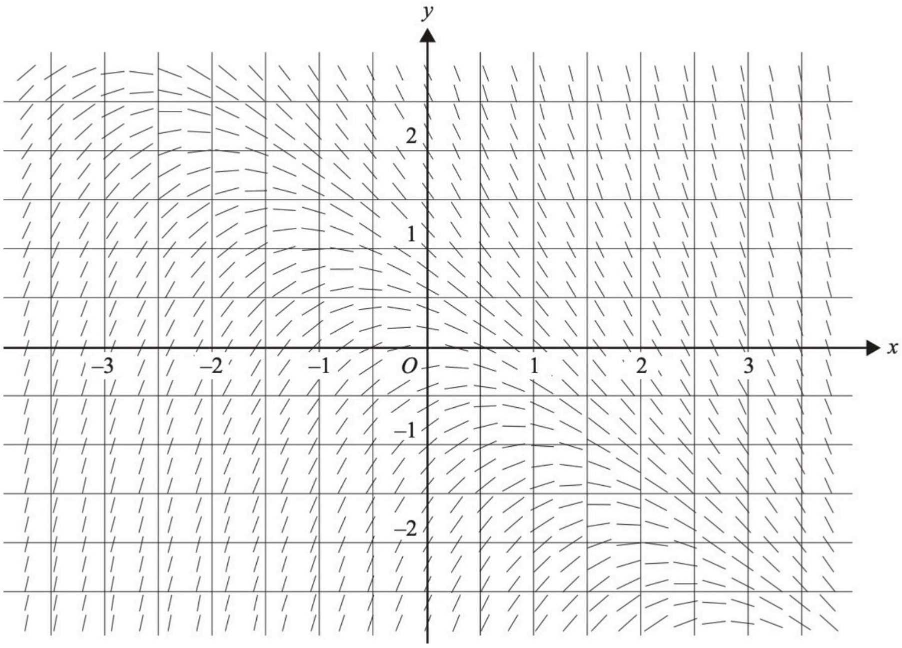
(B) $50\sqrt{2}$ ms^{-1}

(C) $\frac{1}{\sqrt{2}}$ ms^{-1}

(D) $\frac{50}{\sqrt{2}}$ ms^{-1}

Section I continues on next page

5. The direction (slope) field for the differential equation $\frac{dy}{dx} + x + y = 0$ is shown below.



Which point satisfies the solution to the differential equation that includes (0, − 1)?

(A) $(-1.5, -2)$

(B) $(2.5, -1)$

(C) $(3, -1)$

(D) $(3.5, -2.5)$

Section I continues on next page

6. Which of the following is the solution set of the inequation $\dfrac{2}{|x-1|} \le 3$?

(A) $x < 1$ and $x \ge \dfrac{5}{3}$.

(B) $1 < x \le \dfrac{5}{3}$.

(C) $x \le \dfrac{1}{3}$ and $x \ge \dfrac{5}{3}$.

(D) $\dfrac{1}{3} \le x \le \dfrac{5}{3}$.

7. What are the values of p for which $y = e^{px}$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0?$$

(A) $p = -4,\ p = 3$

(B) $p = -3,\ p = 4$

(C) $p = \pm 3$

(D) $p = \pm 4$

8. In a Year 12 Mathematics class, the teacher can give one of 5 grades (A, B, C, D or E) to each student. What is the minimum number of students required so that six students are guaranteed to receive the same grade?

(A) 6

(B) 25

(C) 26

(D) 30

Section I continues on next page

9. In how many different ways can three girls and three boys be seated on a bench for a photograph if the girls must sit apart?

(A) 72

(B) 120

(C) 144

(D) 576

10. Which of the following is equal to $\int_{-1}^{1} \cos^{-1} x \quad dx$?

(A) $\dfrac{\pi}{3}$

(B) $\dfrac{\pi}{2}$

(C) π

(D) 2π

End of Section I

SECTION II

60 marks

Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in a **SEPARATE** writing booklet. Extra writing booklets are available.

In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations. **Marks**

Question 11 (15 marks)

(a) The polynomial $P(x) = 8x^4 - 38x^3 + 9x^2 + ax + b$ has a double root at $x = 3$. **3**

Find the value of a and b, where a and b are real numbers.

(b) Find the exact value of $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$. **2**

(c) Consider the expansion of $(2x - p)^9$. The coefficient of the term in x^6 is $-672\ 000$. **2**

Find the value of p.

Question 11 continues on next page

(d) (i) Write the expression $2\sqrt{3}\sin x + 2\cos x$ in the form $R\sin\left(x+\alpha\right)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. **2**

(ii) Hence find the value of k, where $0 \le k \le 2\pi$, for which **3**

$$\int_0^k (2\sqrt{3}\cos x - 2\sin x)\ dx = 2.$$

(e) Consider the function $f(x) = x^2 - c^2$, where c is a positive real number. **3**

Sketch the graph of $y = \frac{1}{|f(x)|}$, showing all important features including the turning point(s), intercept(s) and asymptote(s).

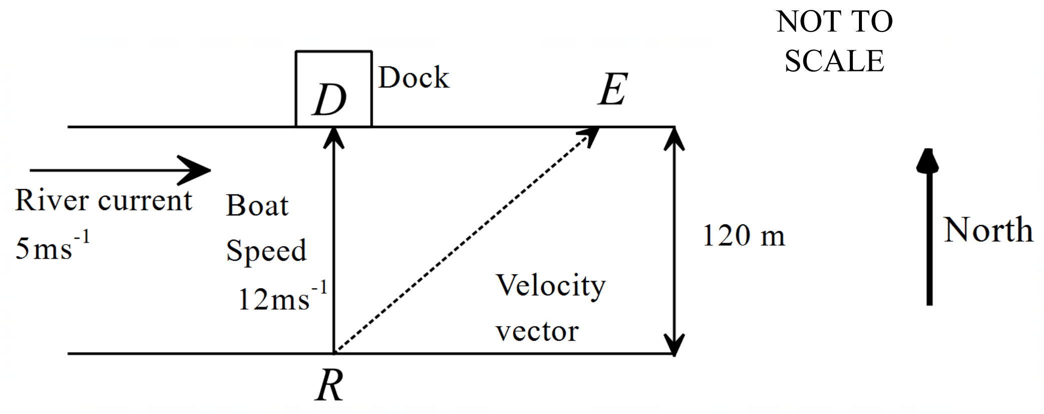
End of Question 11

Question 12 (15 marks) **Start a new writing booklet** **Marks**

(a) Find $\int \frac{\cos^2(\ln x)}{x}\,dx$ using the substitution $u = \ln x$. **3**

(b) Rylie has a boat which moves at a top speed of 12 ms^{-1} in still water.

From point R, he wants to go due north to point D on the opposite side of the river, as shown in the diagram below.



Today the current in the river is flowing at 5 ms^{-1}. From R, he steers the boat due north toward D at top speed. Due to the current he drifts down the river and arrives at point E.

(i) Taking R as the origin, write down Rylie's velocity vector in the form $x\underset{\sim}{i} + y\underset{\sim}{j}$ and find the magnitude of this vector. **2**

(ii) What is the bearing of Rylie's velocity vector and how far does he travel from R to E? **2**

(iii) On what bearing should Rylie have pointed the boat, so that he arrived at D, with the boat travelling at top speed? **2**

Question 12 continues on next page

(c) (i) Show that $\dfrac{1}{(n+1)!} - \dfrac{n+1}{(n+2)!} = \dfrac{1}{(n+2)!}$ **1**

(ii) Use mathematical induction to show that, for all integers $n \geq 1$, **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots\dots\dots\dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

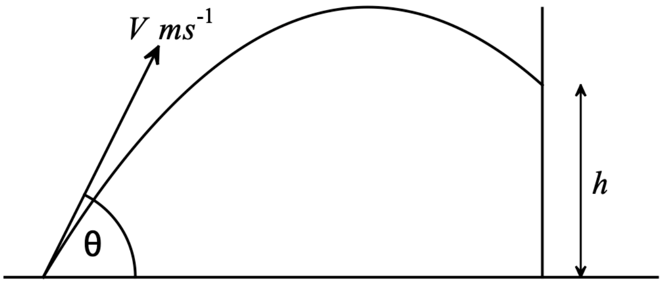
(d) The roots of $x^3 + 4x^2 - 5x + 3 = 0$ are α, β, and γ. **2**

Find the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$.

End of Question 12

Question 13 (15 marks) **Start a new writing booklet** **Marks**

(a) A rocket is fired with a velocity V ms^{-1} at an angle θ to the horizontal from sea level and hits a cliff face as shown. Neglect air resistance and take g = 10ms^{-2}.



NOT TO SCALE

(i) If V = 180 ms^{-1} and θ = 60°, show that at any time t seconds, the position vector of the rocket relative to the point at which it was fired is **2**

$$\underset{\sim}{s}(t) = 90t\underset{\sim}{i} + \left(90\sqrt{3}t - 5t^2\right)\underset{\sim}{j}$$

(ii) Find the maximum height reached by the rocket. **2**

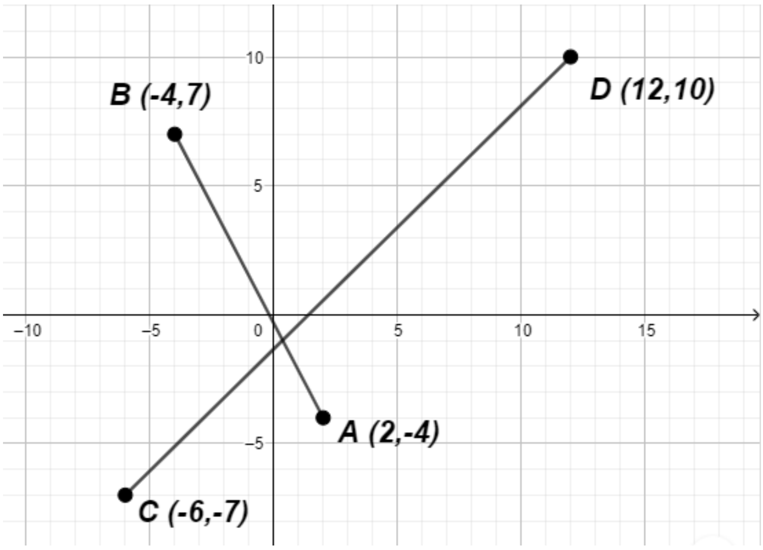
(iii) The base of the cliff is 270 m from the point at which the rocket was fired. Find the value of h. **2**

Question 13 continues on next page

(b) The diagram below shows the lines AB and CD. The line AB passes through the Points $(2, -4)$ and $(-4, 7)$ and the line CD passes through $(-6, -7)$ and $(12, 10)$. **3**

Use vector methods to find the acute angle between the two lines.

Write your answer to the nearest degree.



(c) The proportion, P, of people who know a rumour is modelled by the differential equation $\frac{dP}{dt} = 2P(1-P)$ where t is the time in weeks. Initially only 10% of people know the rumour.

(i) Show that $\frac{1}{P} + \frac{1}{1-P} = \frac{1}{P(1-P)}$. **1**

(ii) Solve the differential equation for P. **3**

(iii) Hence, find the exact time taken for half of the people to know the rumour. **2**

End of Question 13

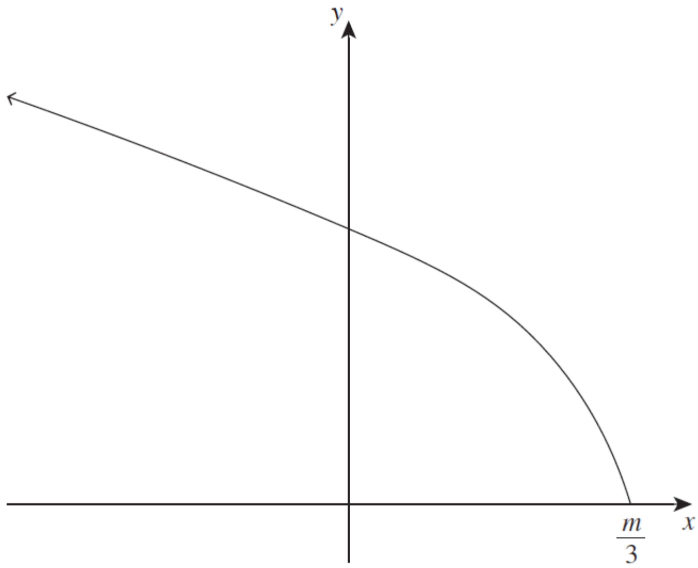
Question 14 (15 marks) **Start a new writing booklet** **Marks**

(a) Find a cartesian equation for the following pair of parametric equations: **2**

$$x = 5\sin\theta$$

$$y = 5\cos\theta + 1$$

(b) Let $f(x) = \sqrt{m - 3x}$ for $x \leq \dfrac{m}{3}$. The graph of $f(x)$ is shown below. **4**



NOT TO SCALE

The area enclosed by the graph $f(x)$, the x-axis and the y-axis is rotated about the y-axis. Find the value of m such that the volume of the solid formed is $\dfrac{5000\pi}{27}$ units^3.

Question 14 continues on next page

(c) The velocity of a particle moving in a straight line is given by $\dot{x} = e^{x} + e^{-x}$, where x in metres is the displacement from the origin. Initially the particle is at the origin.

(i) Show that the time taken by the particle as a function of its displacement is given by **2**

$$t = \int \frac{e^{x}}{e^{2x} + 1}\, dx.$$

(ii) Hence, by using the substitution $u = e^{x}$, prove that. **2**

$$x = \ln\left(\tan\left(t + \frac{\pi}{4}\right)\right).$$

(d) (i) Show that $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\left(\frac{2}{n^2}\right)$ for $n \geq 1$. **2**

(ii) Hence, or otherwise, show that: **3**

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \frac{\pi}{4}.$$

End of Paper

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Ext1 Trial 2023 Solutions

Multiple choice

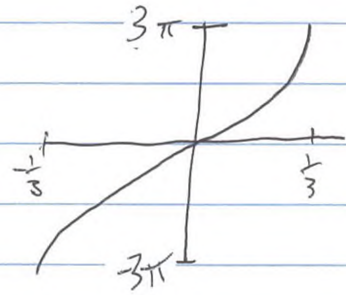
1. $f(x) = \sqrt{x} - 3$ Dom: $x \geq 0$, Range $y \geq -3$

$\therefore f^{-1}(x)$ Domain: $x \geq -3$, Range $y \geq 0$ $\therefore$ (A)

2. $\tan\left(\theta + \frac{\pi}{3}\right) = \dfrac{\tan\theta + \tan\frac{\pi}{3}}{1 - \tan\theta\tan\frac{\pi}{3}}$

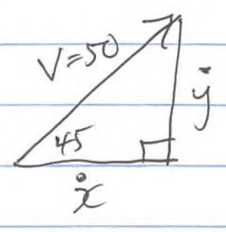
$= \dfrac{\frac{1}{3} + \sqrt{3}}{1 - \frac{\sqrt{3}}{3}} = \dfrac{(1 + 3\sqrt{3})/3}{(3 - \sqrt{3})/3}$ $\therefore$ (C)

3. $y = 6\sin^{-1}(3x)$



$\therefore$ (B)

4.

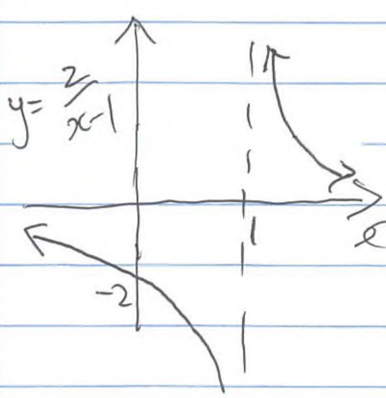


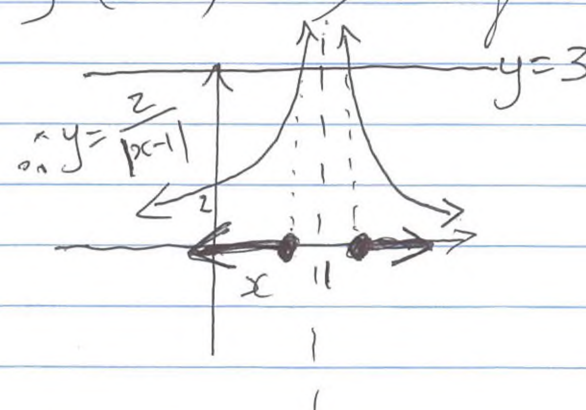
$\frac{1}{\sqrt{2}} = \frac{\dot{x}}{50}$

$\therefore \dot{x} = \frac{50}{\sqrt{2}}$ $\therefore$ (D)

5. By inspection, only $(3.5, -2.5)$ satisfies the DE $\therefore$ (D)

6.





$\therefore$ (C)

7. $y' = p e^{px}$

$y'' = p^2 e^{px}$

i.e $p^2 e^{px} - p e^{px} - 12 e^{px} = 0$

$e^{px}(p^2 - p - 12) = 0$

$e^{px}(p-4)(p+3) = 0$ $\therefore$ (B)

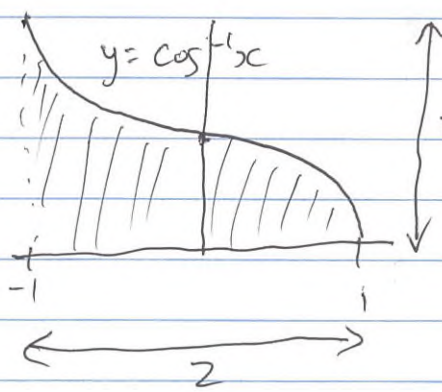
8. $5A's + 5B's + 5C's + 5D's + 5E's + 1$

$= 26$ $\therefore$ (C)

9. $B_1\, B_2\, B_3 \rightarrow 3!$ ways to arrange the group of boys.

i.e $\_\, B_1 \_\, B_2 \_\, B_3 \_$ $4!$ ways to arrange the girls.

$\therefore\ 3! \times 4!$ $\therefore$ (C)

10. 

$\int_{-1}^{1} \cos^{-1} x \, dx = \frac{1}{2} \cdot 2\pi$ $\therefore$ (C)

Question 11

(a) $P(x) = 8x^4 - 38x^3 + 9x^2 + ax + b$
$P'(x) = 32x^3 - 114x^2 + 18x + a$

now $P(3) = 648 - 1026 + 81 + 3a + b = 0$
i.e $3a + b = 297$

and $P'(3) = 864 - 1026 + 54 + a = 0$
$\underline{a = 108}$

$\therefore 3(108) + b = 297$
$\underline{b = -27}$

(b) $$\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{2^2 - x^2}}\, dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

(c) $$(2x - p)^9 = \sum_{k=0}^{9} \binom{9}{k} (2x)^k (-p)^{9-k}$$

term in $x^6 = \binom{9}{6}(2x)^6(-p)^3$

i.e $-672000 = -84 \cdot 64 \cdot p^3$

$p^3 = 125$
$p = 5$

d(i) $2\sqrt{3}\sin x + 2\cos x = R\sin x\cos\alpha + R\sin\alpha\cos x$

$$R\cos\alpha = 2\sqrt{3}$$
$$R\sin\alpha = 2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = (4\times 3) + 4$$
$$\therefore R = 4 \qquad (R>0)$$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{2\sqrt{3}} = \tan\alpha$$

$$\text{i.e} \quad \tan\alpha = \frac{1}{\sqrt{3}}$$
$$\therefore \alpha = \frac{\pi}{6} \qquad \left(0<\alpha<\frac{\pi}{2}\right)$$

i.e $2\sqrt{3}\sin x + 2\cos x = 4\sin\left(x+\frac{\pi}{6}\right)$

(ii) $\displaystyle\int_0^k (2\sqrt{3}\cos x - 2\sin x)\,dx$

$$= \left[2\sqrt{3}\sin x + 2\cos x\right]_0^k = 2$$

i.e $2\sqrt{3}\sin k + 2\cos k - (2\sqrt{3}\sin 0 + 2\cos 0) = 2$

$$2\sqrt{3}\sin k + 2\cos k - 2 = 2$$

i.e $2\sqrt{3}\sin k + 2\cos k = 4$

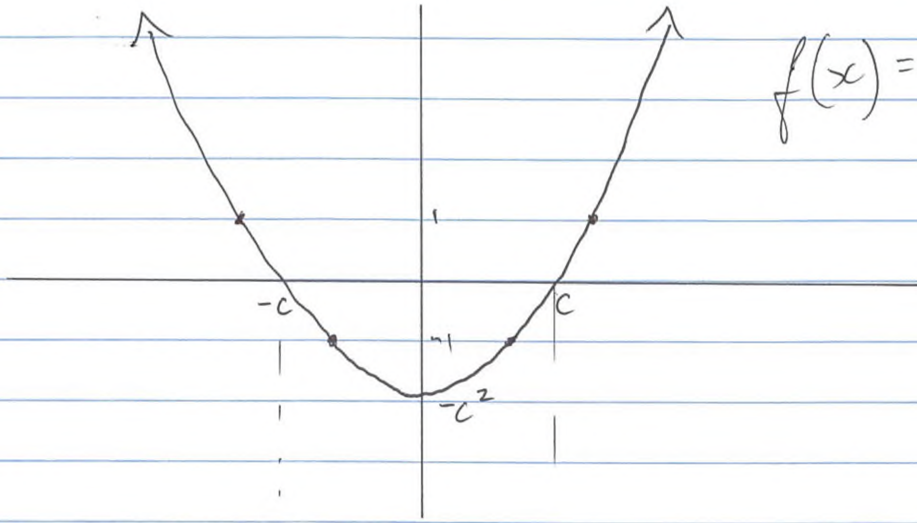
$\therefore$ using (i)

$$4\sin\left(k+\frac{\pi}{6}\right) = 4$$
$$\sin\left(k+\frac{\pi}{6}\right) = 1$$
$$k+\frac{\pi}{6} = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$$

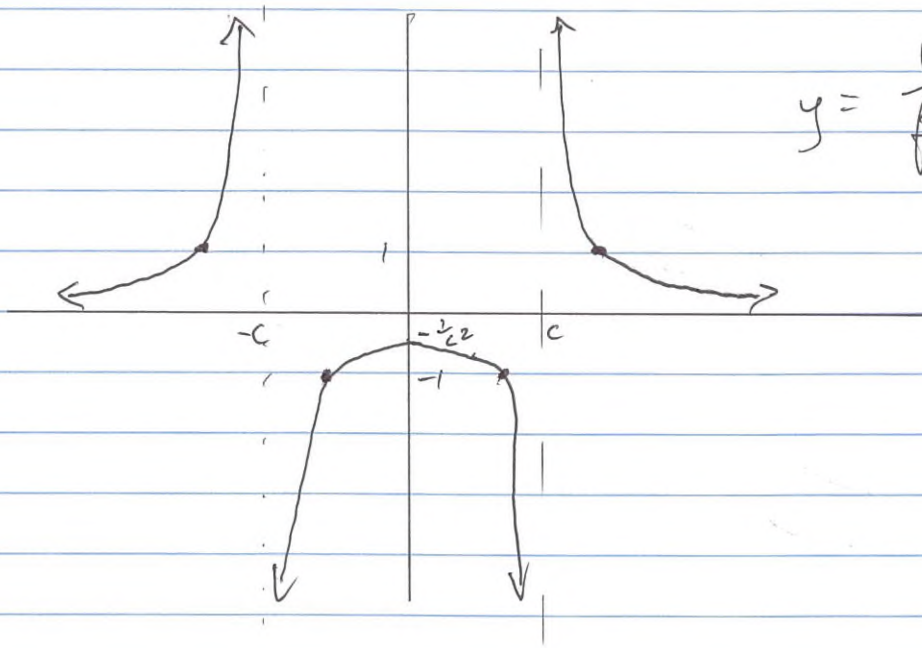
$$\therefore k = \frac{\pi}{3}$$

1 solution only for $0 \le k \le 2\pi$

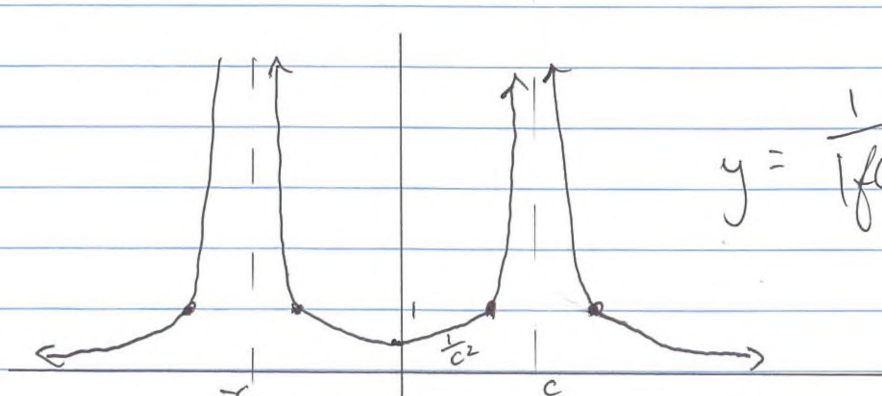
(e)



$f(x) = x^2 - c^2$



$y = \frac{1}{f(x)}$



$y = \frac{1}{|f(x)|}$

asymptotes @ $x = \pm c$

turning point and y-intercept @ $\left(0, \frac{1}{c^2}\right)$

Question 12.

(a) $I = \displaystyle\int \frac{\cos^2(\ln x)}{x}\, dx$

$u = \ln x$

$du = \frac{1}{x}\, dx$

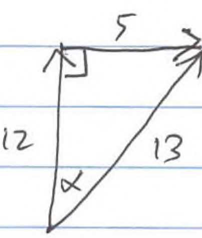
$\therefore I = \displaystyle\int \cos^2(\ln x)\, \frac{dx}{x} = \int \cos^2 u \, du$

$= \frac{1}{2}\displaystyle\int 1 + \cos 2u \, du$

$= \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]$

$\therefore I = \frac{1}{2}\left[\ln x + \frac{1}{2}\sin(2\ln x)\right] + C$

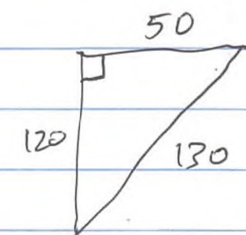
(b) (i) $\underset{\sim}{v} = 5\underset{\sim}{i} + 12\underset{\sim}{j}$



Using Pythagoras', $|\underset{\sim}{v}| = 13\,m/s$

(ii) From the diagram, $\tan\alpha = 5/12$

$\therefore \alpha = 22.61\ldots^\circ$



$\therefore$ Rylie's bearing is 022.6° and he travels 130m.

(iii) using the triangle:

$\sin\theta = 5/12$

$\therefore \theta = 24.62\ldots^\circ$

i.e Rylie should use a bearing of $360 - 24.62$

$\doteqdot 335^\circ$

12(c)(i) $LHS = \frac{1}{(n+1)!} - \frac{n+1}{(n+2)!}$

$$= \frac{(n+2) - (n+1)}{(n+2)!}$$

$$= \frac{1}{(n+2)!} = RHS.$$

(ii) Prove $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for $n \geq 1$

Test $n=1$

$LHS = \frac{1}{(1+1)!} = \frac{1}{2!} = \frac{1}{2}$

$RHS = 1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2!} = \frac{1}{2} = LHS.$

$\therefore$ true for $n=1$

Assume true for $n=k$, i.e assume:

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

Now show true for $n=k+1$, ie show:

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

$LHS = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$ using the assumption

$$= 1 - \left(\frac{1}{(k+1)!} - \frac{k+1}{(k+2)!}\right)$$

$= 1 - \frac{1}{(k+2)!}$ using part (i)

$= RHS.$

(d) $\overset{-}{x^3} + \overset{+}{4x^2} - \overset{-}{5x} + 3 = 0$

Now $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

$= \frac{\gamma}{\alpha\beta\gamma} + \frac{\beta}{\alpha\beta\gamma} + \frac{\alpha}{\alpha\beta\gamma}$

$= \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma}$

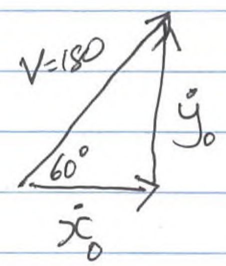
$= \frac{-b/a}{-d/a}$

$= -4/-3$

$= 4/3$

Question 13

a (i)



initial velocity components:

$\dot{x}_0 = 180\cos 60$
$= 90\ m/s$

$\dot{y}_0 = 180\sin 60$
$= 90\sqrt{3}\ m/s.$

horizontal motion:

$\ddot{x} = 0$

$\dot{x} = \int 0\, dt$
$= C$

at $t=0$, $\dot{x} = 90$

$\therefore x = \int_0^t 90\, dt$
$= [90t]_0^t$
$= 90t - 0$

$x = 90t$

vertical motion.

$\ddot{y} = -10$

$\dot{y} = \int -10\, dt$
$= -10t + C$

at $t=0$, $\dot{y} = 90\sqrt{3}$

$\therefore \dot{y} = -10t + 90\sqrt{3}$

$\therefore y = \int -10t + 90\sqrt{3}\, dt$

$= -5t^2 + 90\sqrt{3}t + C$

at $t=0$, $y=0$, $\therefore C=0$

i.e $y = -5t^2 + 90\sqrt{3}t$

$\therefore \underset{\sim}{s}(t) = 90t\,\underset{\sim}{i} + (90\sqrt{3}t - 5t^2)\underset{\sim}{j}$

(ii) Maximum height occurs when $\dot{y} = 0$
i.e when $90\sqrt{3} - 10t = 0$
$9\sqrt{3} - t = 0$
$t = 9\sqrt{3}$ sec

sub $t = 9\sqrt{3}$ into y:
$y = -5(9\sqrt{3})^2 + 90\sqrt{3}(9\sqrt{3})$
$= 1215$ m.

(iii) $x = 270 = 90t$
$\therefore$ at $t = 3$ sec (sub this into y)
$y = -5(3)^2 + 90\sqrt{3}(3)$
$= (-45 + 270\sqrt{3})$ m

13(b) $\overrightarrow{BA} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \end{pmatrix}$ $\therefore$ $|\overrightarrow{BA}| = \sqrt{6^2+11^2} = \sqrt{157}$

$\overrightarrow{DC} = \begin{pmatrix} -6 \\ -7 \end{pmatrix} - \begin{pmatrix} 12 \\ 10 \end{pmatrix} = \begin{pmatrix} -18 \\ -17 \end{pmatrix}$ $\therefore$ $|\overrightarrow{DC}| = \sqrt{18^2+17^2} = \sqrt{613}$

$$\overrightarrow{BA} \cdot \overrightarrow{DC} = \begin{pmatrix} 6 \\ -11 \end{pmatrix} \cdot \begin{pmatrix} -18 \\ -17 \end{pmatrix}$$

$$= 6 \times (-18) + (-11) \cdot (-17)$$

$$= 79$$

and

$$\overrightarrow{BA} \cdot \overrightarrow{DC} = |\overrightarrow{BA}| \times |\overrightarrow{DC}| \cos\theta$$

ie

$$79 = \sqrt{157} \cdot \sqrt{613} \cos\theta$$

$$\therefore \cos\theta = \frac{79}{\sqrt{157} \cdot \sqrt{613}}$$

$$\theta = 75.247\ldots$$

i.e $\theta = 75^\circ$ (to nearest degree).

13(c) (i) $LHS = \frac{1}{P} + \frac{1}{1-P}$

$= \frac{1-P+P}{P(1-P)} = RHS.$

(ii) $\frac{dP}{dt} = 2P(1-P)$ Separate variables:

$$\int_{0.1}^{P} \frac{dP}{P(1-P)} = 2\int_0^t dt$$

using part (i)

$$\int_{0.1}^{P} \frac{1}{P} + \frac{1}{1-P}\, dP = 2\left[t\right]_0^t$$

$$\left[\ln P - \ln(1-P)\right]_{0.1}^{P} = 2t - 0$$

$$\ln P - \ln(1-P) - (\ln(0.1) - \ln(0.9)) = 2t$$

$$\ln \frac{P}{1-P} + \ln \frac{9}{1} = 2t$$

i.e

$$2t = \ln \frac{9P}{1-P}$$

$$e^{2t} = \frac{9P}{1-P}$$

$$(1-P)e^{2t} = 9P$$

$$e^{2t} = 9P + Pe^{2t}$$

$$= P(9 + e^{2t})$$

$$\therefore P = \frac{e^{2t}}{9+e^{2t}} \cdot \frac{e^{-2t}}{e^{-2t}} = \frac{1}{9e^{-2t}+1}$$

(iii) When $P = \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{9e^{-2t}+1}$$

$$\therefore 2 = 9e^{-2t}+1$$

$$e^{-2t} = \frac{1}{9}$$

$$e^{2t} = 9 \quad \therefore t = \frac{1}{2}\ln 9$$

Question 14

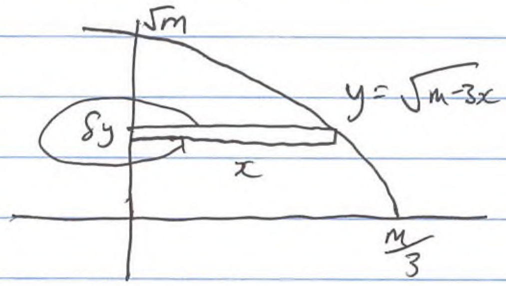
(a) $\sin\theta = \frac{x}{5}$, $\cos\theta = \frac{y-1}{5}$

$\therefore \sin^2\theta + \cos^2\theta = 1$

i.e $\frac{x^2}{25} + \frac{(y-1)^2}{25} = 1$

alternatively $x^2 + (y-1)^2 = 25$

(b)



$y = \sqrt{m-3x}$

$y^2 = m-3x$

$x = \frac{m-y^2}{3}$

$\therefore x^2 = \frac{m^2-2my^2+y^4}{9}$

typical slice rotated will have $V = \pi x^2 \delta y$

$\therefore \quad V = \pi \int_0^{\sqrt{m}} x^2 \, dy$

$= \frac{\pi}{9} \int_0^{\sqrt{m}} m^2 - 2my^2 + y^4 \, dy$

$= \frac{\pi}{9} \left[m^2 y - \frac{2my^3}{3} + \frac{y^5}{5} \right]_0^{\sqrt{m}}$

$= \frac{\pi}{9} \left[m^{2.5} - \frac{2}{3} m^{2.5} + \frac{1}{5} m^{2.5} - 0 \right]$

i.e

$\frac{5000\pi}{27} = \frac{\pi}{9} \left(\frac{8}{15} m^{5/2} \right)$

$m^{5/2} = \frac{5000}{27} \cdot \frac{9 \cdot 15}{8}$

$m^{5/2} = 3125$

$\therefore m = 25$

(c) (i) $\dot{x} = \frac{dx}{dt} = e^{x} + e^{-x}$

$$= e^{x} + \frac{1}{e^{x}} = \frac{e^{2x}+1}{e^{x}}$$

$$\therefore \frac{dt}{dx} = \frac{e^{x}}{e^{2x}+1}$$

$$\therefore \int_0^t dt = \int_0^x \frac{e^{x}}{e^{2x}+1}\,dx$$

(ii) Let $u = e^{x}$

$du = e^{x}dx$ when $x = 0$, $u = 1$

$x = x$, $u = e^{x}$

$$\therefore t = \int_1^{e^x} \frac{du}{u^2+1}$$

$$= \left[\tan^{-1} u\right]_1^{e^x}$$

$$= \tan^{-1} e^{x} - \tan^{-1} 1$$

$$\therefore t = \tan^{-1} e^{x} - \frac{\pi}{4}$$

$$\left(t + \frac{\pi}{4}\right) = \tan^{-1} e^{x}$$

$$e^{x} = \tan\left(t + \frac{\pi}{4}\right)$$

$$\therefore x = \ln\left[\tan\left(t + \frac{\pi}{4}\right)\right]$$

(d) (i) Show $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\left(\frac{2}{n^2}\right) \quad n \geq 1$

now,

$$\tan\left[\tan^{-1}(n+1) - \tan^{-1}(n-1)\right]$$

$$= \frac{\tan\left[\tan^{-1}(n+1)\right] - \tan\left[\tan^{-1}(n-1)\right]}{1 + \tan\left[\tan^{-1}(n+1)\right]\tan\left[\tan^{-1}(n-1)\right]}$$

$$= \frac{n+1 - (n-1)}{1 + (n+1)(n-1)}$$

$$= \frac{2}{n^2}$$

ie $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\left(\frac{2}{n^2}\right)$.

(ii) Hence show $\sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \frac{\pi}{4}$

$LHS = \sum_{r=1}^{n}\left[\tan^{-1}(k+1) - \tan^{-1}(k-1)\right]$ using part (i)

$= \tan^{-1}2 - \tan^{-1}0 + \tan^{-1}3 - \tan^{-1}1 + \tan^{-1}4 - \tan^{-1}2 + \tan^{-1}5 - \tan^{-1}3 + \cdots$

$+ \cdots + \tan(n-1) - \tan(n-3) + \tan^{-1}(n) - \tan^{-1}(n-2) + \tan^{-1}(n+1) - \tan^{-1}(n-1)$

Rearranging:

$= \left[\tan^{-1}2 + \tan^{-1}3 + \tan^{-1}4 + \cdots + \tan^{-1}(n-1)\right] + \tan^{-1}(n) + \tan^{-1}(n+1)$

$- \left[\tan^{-1}(0) + \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 + \cdots + \tan^{-1}(n-2) + \tan^{-1}(n-1)\right]$

$\therefore$ Remaining terms after cancelling $= \tan^{-1}(n) + \tan^{-1}(n+1) - \tan^{-1}1$

Now, similar to part (i):

$$\tan\left[\tan^{-1}(n) + \tan^{-1}(n+1)\right] = \frac{n+n+1}{1-n(n+1)} = \frac{2n+1}{1-n-n^2}$$

i.e $\tan^{-1}(n) + \tan^{-1}(n+1) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$

$$\therefore \sum_{r=1}^{n}\tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \tan^{-1}1$$

$$= \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \frac{\pi}{4}.$$

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \frac{\pi}{4}$$

By induction

Step 1: Let $n=1$

$$LHS = \sum_{r=1}^{1} \tan^{-1}\left(\frac{2}{r}\right)$$

$$= \tan^{-1}(2)$$

$$RHS = \tan^{-1}\left(\frac{2(1)+1}{1-1-(1)^2}\right) - \frac{\pi}{4}$$

$$= \tan^{-1}(-3) - \frac{\pi}{4}.$$

now $\tan\left(\tan^{-1}(-3) - \frac{\pi}{4}\right) = \dfrac{\tan(\tan^{-1}(-3)) - \tan \pi/4}{1 + \tan(\tan^{-1}(-3)) \tan \pi/4}$

$$= \frac{-3 - 1}{1 + -3 \times 1}$$

$$= 2.$$

$\therefore \tan^{-1}(-3) - \pi/4 = \tan^{-1}(2)$

$= \cancel{RH}\ LHS.$

Step 2: assume true for $n=k$.

i.e. $\displaystyle\sum_{k=1}^{k} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2k+1}{1-k-k^2}\right) - \frac{\pi}{4}.$

Step 3: Prove for $n=k+1$

i.e. $\displaystyle\sum_{r=1}^{k+1} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2(k+1)+1}{1-(k+1)-(k+1)^2}\right)$

$$= \tan^{-1}\left(\frac{2k+3}{-k^2-3k-1}\right)$$

$$LHS = \sum_{r=1}^{k} \tan^{-1}\left(\frac{2}{r^2}\right) + \tan^{-1}\left(\frac{2}{(k+1)^2}\right)$$

$= \tan^{-1}\left(\dfrac{2k+1}{1-k-k^2}\right) + \tan^{-1}\left(\dfrac{2}{(k+1)^2}\right)$ from assumption.

So $\tan(LHS) = \dfrac{\tan\left(\tan^{-1}\left(\frac{2k+1}{1-k-k^2}\right)\right) + \tan\left(\tan^{-1}\left(\frac{2}{(k+1)^2}\right)\right)}{1 - \tan\left(\tan^{-1}\left(\frac{2k+1}{1-k-k^2}\right)\right)\tan\left(\tan^{-1}\left(\frac{2}{(k+1)^2}\right)\right)}$

$$= \frac{\frac{2k+1}{1-k-k^2} + \frac{2}{(k+1)^2}}{1 - \frac{2k+1}{1-k-k^2} \cdot \frac{2}{(k+1)^2}}$$

$$= \frac{(2k+1)(k+1)^2 + 2(1-k-k^2)}{(1-k-k^2)(k+1)^2 - 2(2k+1)}$$

$$= \frac{2k^3+4k^2+2k+k^2+2k+1+2-2k-2k^2}{k^2+2k+1-k^2-2k^2-k-k^4-2k^3-k^2-4k-2}$$

$$= \frac{2k^3+3k^2+2k+3}{-k^4-3k^3-2k^2-3k-1}$$

$$= \frac{(2k+3)(k^2+1)}{-(k^2+1)(k^2+3k+1)}$$

as $2k^3+3k^2+2k+3 = (2k+3)k^2 + (2k+3)$
$= (2k+3)(k^2+1)$

&

$$\begin{array}{r|l} & k^2+3k+1 \\ \hline k^2+1 & k^4+3k^3+2k^2+3k+1 \\ & k^4 + k^2 \\ \hline & 3k^3+k^2+3k \\ & 3k^3 + 3k \\ \hline & k^2 + 1 \\ & k^2 + 1 \end{array}$$

$$= \frac{2k+3}{-k^2-3k-1}$$

$\therefore \quad LHS = \tan^{-1} \dfrac{2k+3}{-k^2-3k-1}$

$= RHS$

Step 4: So proof holds by the inductive process